

What kind of laws come from urns?

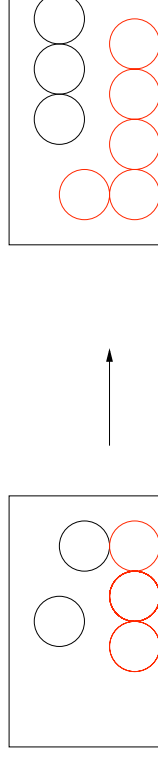
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Pólya-Eggenberger urns with 2 colours



Replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c, d integers.

Composition vector $U^{DT}(n) = \begin{pmatrix} \# \text{ red at time } n \\ \# \text{ black at time } n \end{pmatrix}$; $U^{DT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

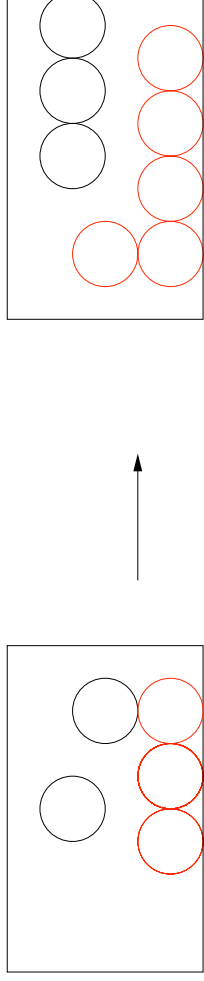
Question: Asymptotics of $U^{DT}(n)$?

References

- [1] G. PÓLYA Sur quelques points de la théorie des probabilités. *Ann. Inst. Henri Poincaré* **1** (1931), 117–161.
- [2] H.M. MAHMOUD Pólya urn models. CRC Press, (2008)
- [3] S. JANSON Functional limit theorem for multitype branching processes and generalized Pólya urns. *Stochastic Processes and Applications*, **110** (2004), 177–245.
- [4] P. FLAJOLET, J. GABARRÓ, H. PEKARI Analytic urns. *Annals of Probability*, **33** (3) (2005), 1200–1233.
- [5] N. POUYANNE An algebraic approach of Pólya processes (2005). *Ann. Inst. Henri Poincaré*, Vol. 44, No. 2, 293–323.

Pólya-Eggenberger urns with 2 colours

One urn, red and black balls



Replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c, d integers.

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Assumption

Balance: $a + b = c + d := S$

at time $n = x_0 + y_0 + Sn$.

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} S & 0 \\ 0 & m \end{pmatrix} \quad m = \sigma S = a - c = d - b$$

2 kinds of asymptotics:

- Gaussian: when $\sigma S < \frac{1}{2}S$ “small” urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

- non Gaussian: when $\frac{1}{2}S < \sigma S < S$ “large” urn

$$U^{DT}(n) = nv_1 + n^\sigma W^{DT} v_2 + o(n^\sigma)$$

where

v_1, v_2 are deterministic vectors

W^{DT} is a martingale’s limit

$o(\cdot)$ means a.s. and in any $L^p, p \geq 1$

the moments of W^{DT} can be recursively calculated.

Goal: distribution of W^{DT}

Embedding

Continuous-time Markov process $(U^{CT}(t))_{t \geq 0}$

$$U^{CT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

$\mathcal{Exp}(1)$ clock on each ball; independence of the clocks.

Branching:

- if a red ball rings, replace it by $a + 1$ red balls and b black balls
- if a black ball rings, replace it by c red balls and $d + 1$ black balls

$0 < \tau_1 < \dots < \tau_n < \dots$ are the jump times

$$(U^{CT}(\tau_n))_{n \geq 0} \stackrel{\mathcal{D}}{=} (U^{DT}(n))_{n \geq 0}$$

Benefit: **independence**

Outline

Discrete urn $U^{DT}(n)$ $\longrightarrow$ Continuous-time branching process $U^{CT}(t)$
 W^{DT} W^{CT}

connection: $W^{CT} = \xi^\sigma W^{DT}$

where $\xi \stackrel{\mathcal{D}}{=} \Gamma\left(\frac{x_0+y_0}{S}\right)$

$\downarrow$
 Independence properties

$\downarrow$
 Dislocation equations on W^{CT}

$\downarrow$
 Differential system on the Fourier transform of W^{CT}

$\downarrow$
 Resolution $\longrightarrow$ distribution of W^{CT}

(connection) $\downarrow$

Distribution of W^{DT}

Dislocation equations

Simplification of the notations:

$$\begin{aligned} \textcolor{red}{X}_t &:= U^{CT}(t) \text{ starting from } (x_0, y_0) = (1, 0) \\ \textcolor{red}{Y}_t &:= U^{CT}(t) \text{ starting from } (x_0, y_0) = (0, 1) \end{aligned}$$

$\forall t > \text{first splitting time } \tau_1,$

$$\begin{cases} X_t \stackrel{\mathcal{D}}{=} [a+1]X_{t-\tau_1} + [b]Y_{t-\tau_1} \\ Y_t \stackrel{\mathcal{D}}{=} [c]X_{t-\tau_1} + [d+1]Y_{t-\tau_1}, \end{cases}$$

$[\]$ is a notation

Take projections, renormalize, take the limit, $t \rightarrow +\infty$:

$$\begin{cases} W_{(1,0)}^{CT} \stackrel{not}{=} X \stackrel{\mathcal{D}}{=} e^{-m\tau_1} \left([a+1]X + [b]Y \right) \\ W_{(0,1)}^{CT} \stackrel{not}{=} Y \stackrel{\mathcal{D}}{=} e^{-m\tau_1} \left([c]X + [d+1]Y \right) \end{cases}$$

Goal: ? distribution of $\textcolor{red}{X}$ and $\textcolor{red}{Y}$;

1. If you love **Analytic Combinatorics**:

$$\text{Let } a_n = \mathbb{E}(X^n) \quad b_n = \mathbb{E}(Y^n)$$

$$\text{Let } F(T) := \sum_{n \geq 0} \frac{a_n}{n!} T^n \quad G(T) := \sum_{n \geq 0} \frac{b_n}{n!} T^n \quad \text{Laplace series}$$

Dislocation equations

↓

Recursion on the moments

↓ (multinomial formula)

$$\left\{ \begin{array}{l} F(T) + mTF'(T) = F(T)^{a+1}G(T)^b \\ G(T) + mTG'(T) = F(T)^cG(T)^{d+1} \end{array} \right\} + \text{initial conditions}$$

2. If you like Probability, too:

$$\text{Let } \mathcal{F}(x) = \mathbb{E}(e^{ixX}) \quad \mathcal{G}(x) = \mathbb{E}(e^{ixY})$$

Insert in the dislocation equation:

$$\mathcal{F}(x) = \frac{x}{mx^{1+\frac{1}{m}}} \int_0^x \mathcal{F}^{a+1}(t) \mathcal{G}^b(t) \frac{dt}{t^{1-1/m}}, \quad (x > 0)$$

Derive:

$$\begin{cases} \mathcal{F}(x) + mx\mathcal{F}'(x) = \mathcal{F}(x)^{a+1}\mathcal{G}(x)^b \\ \mathcal{G}(x) + mx\mathcal{G}'(x) = \mathcal{F}(x)^c\mathcal{G}(x)^{d+1} \end{cases} + \text{initial conditions}$$

Singular in 0

Resolution of the differential system

$$\begin{cases} \mathcal{F}(x) + mx\mathcal{F}'(x) = \mathcal{F}(x)^{a+1}\mathcal{G}(x)^b \\ \mathcal{G}(x) + mx\mathcal{G}'(x) = \mathcal{F}(x)^c\mathcal{G}(x)^{d+1} \end{cases}$$

Change of function, with $x^S(-Sw)^m = 1$.

$$\begin{cases} f(w) = x^{\frac{1}{m}}\mathcal{F}(x) \\ g(w) = x^{\frac{1}{m}}\mathcal{G}(x), \end{cases}$$

Non singular differential system

$$(pf60) \quad \begin{cases} f' = f^{a+1}g^b \\ g' = f^cg^{d+1} \end{cases}$$

Resolution of the differential system

$$(pf60) \quad \left\{ \begin{array}{l} f' = f^{a+1} g^b \\ g' = f^c g^{d+1} \end{array} \right.$$

First integral:

$$\frac{1}{g^m} - \frac{1}{f^m} = Constant = \frac{1}{\kappa^m}$$

g is a function of f :

$$g = \frac{f}{\left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{1/m}}$$

Insert in (pf60):

$$\frac{f'}{f^{S+1}} \times \left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{\frac{b}{m}} = 1 \quad (+ \text{ boundary condition})$$

Resolution of the differential system $(\dots) \frac{f'}{f^{S+1}} \times \left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{\frac{b}{m}} = 1$
 can be written

$$\frac{d}{dw} \left(I \circ \left(\frac{f}{\kappa} \right) \right) = -\kappa^S \quad \text{with} \quad I'(z) = -\frac{(1+z^m)^{\frac{b}{m}}}{z^{S+1}}.$$

and

$$I(z) = \int_{[z,z\infty)} (1+u^m)^{\frac{b}{m}} \frac{du}{u^{S+1}}$$

$$I\left(\frac{f(w)}{\kappa}\right) = C_0 - \kappa^S w, \quad \text{with } C_0 \text{ a negative real constant.}$$

Let J be the inverse function of I

$$f(w) = \kappa J\left(C_0 - \kappa^S w\right)$$

$$\mathcal{F}(x) = \kappa x^{-\frac{1}{m}} J\left(C_0 + \frac{\kappa^S}{S} x^{-\frac{S}{m}}\right)$$

Consequences of

$$\mathcal{F}(x) = \kappa x^{-\frac{1}{m}} J \left(C_0 + \frac{\kappa^S}{S} x^{-\frac{S}{m}} \right)$$

many...

Ex 1:

$$\mathcal{F}(x) \underset{x \rightarrow +\infty}{\sim} \kappa J(C_0) x^{-\frac{1}{m}}$$

The Fourier transform of X is not integrable. The existence of a density does not come from there.

Ex 2:

X is not stable

PROOF. Insert the Fourier transform of a stable distribution in the differential equation and see the contradiction.

More on these distributions:

Define more general continuous-time $\mathbb{R}^2$ -valued Markov processes.

Initial condition: $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$

Activities α and jumps governed by matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

State vector $U_{x_0, y_0, \alpha, R}(t)$,

renormalized $\rightarrow$ martingale

$$\xrightarrow[t \rightarrow \infty]{} W_{x_0, y_0, \alpha, R} \text{ with the property:}$$

for every $n \in \mathbb{N}$,

$$W_{x_0, y_0, \alpha, R} = [n] W_{\frac{x_0}{n}, \frac{y_0}{n}, \alpha, R}.$$

W is infinitely divisible

Remember:

$$W^{CT} = \xi^\sigma W^{DT}$$

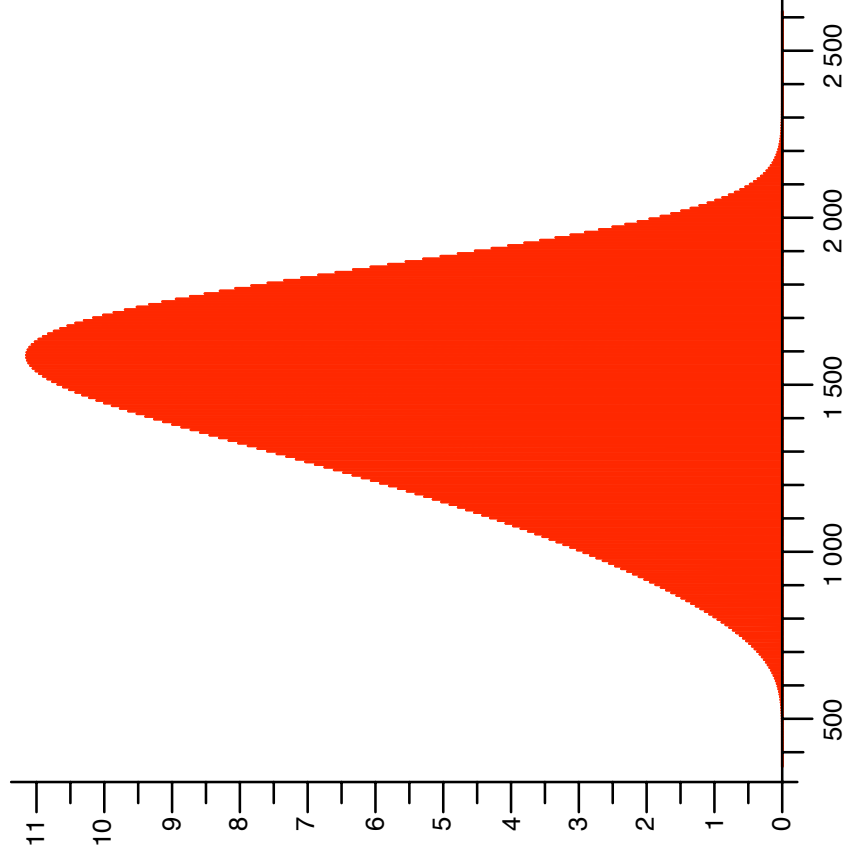
$\uparrow$ unimodal $\uparrow$ non unimodal

Replacement matrix $R = \begin{pmatrix} 9 & 2 \\ 1 & 10 \end{pmatrix}$

Starting from 1 red ball and 0 black ball;

Figure = W^{DT}

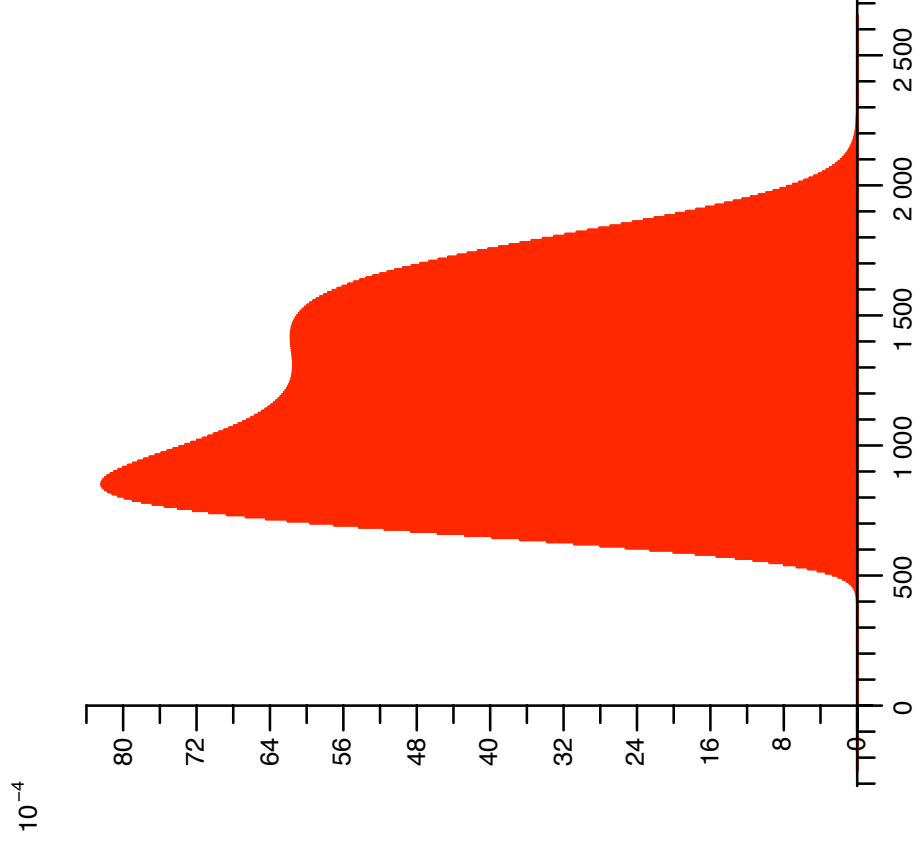
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Replacement matrix $R = \begin{pmatrix} 9 & 2 \\ 1 & 10 \end{pmatrix}$

Starting from 1 red ball and 1 black ball;

Figure = W^{DT}



Replacement matrix $R = \begin{pmatrix} 6 & 1 \\ 2 & 5 \end{pmatrix}$

Starting from 9 red balls and 9 black balls;

Figure = W^{DT}

