Uncommon Suffix Tries

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Abstract

Common assumptions on the source producing the words inserted in a suffix trie with n leaves lead to a $\ln n$ height and saturation level. We provide an example of a suffix trie whose height increases faster than a power of n and another one whose saturation level is negligible with respect to $\ln n$. Both are built from VLMC (Variable Length Markov Chain) probabilistic sources and are easily extended to families of tries having the same properties. The first example corresponds to a "logarithmic infinite comb" and enjoys a non uniform polynomial mixing. The second one corresponds to a "factorial infinite comb" for which mixing is uniform and exponential.

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1 Introduction

Trie (abbreviation of re*trie*val) is a natural data structure, efficient for searching words in a given set and used in many algorithms as data compression, spell checking or IP addresses lookup. A *trie* is a digital tree in which words are

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inserted in external nodes. The trie process grows up by successively inserting words according to their prefixes. A precise definition will be given in Section 4.1. As soon as a set of words is given, the way they are inserted in the trie is deterministic. Nevertheless, a trie becomes random when the words are randomly drawn: each word is produced by a probabilistic source and n words are chosen (usually independently) to be inserted in a trie. A *suffix trie* is a trie built on the suffixes of *one* infinite word. The randomness then comes from the source producing such an infinite word and the successive words inserted in the tree are far from being independent, they are strongly correlated.
Suffix tries, also called suffix trees in the literature, have been developped with tools from analysis of algorithms and information theory on one side and from probability theory and ergodic theory on the other side. Construction algorithms of such trees go back to Weiner [17] in 1973 and many applications in computer science and biology can be found in Gusfield's book [9]. As a major application of suffix tries one can cite an efficient implementation of the famous Lempel-Ziv lossless compression algorithm LZ77. The first results on the height of random suffix trees are due to Szpankowski [15] and Devroye *et al.* [3], and the most recent results are in Szpankowski [16] and Fayolle [5].
The growing of suffix tries (called prefix trees in Shields' book [14]) is closely related to second occurrences of patterns; this will be precisely explained later in Section 4.1. Consequently, many results on this subject can be found in papers dealing with occurrences of words, renewal theory or waiting times like Shields [13] or Wyner-Ziv [18].
In the present paper we are interested in the height H_n and the saturation level ℓ_n of a suffix trie $\mathcal{T}_n$ containing the first n suffixes of an infinite word produced by a probabilistic source. The analysis of the height and the saturation level is usually motivated by optimization of the memory cost. Height is clearly relevant to this point; saturation level is algorithmically relevant as well because internal nodes below the saturation level are often replaced by a less expansive table. Such parameters depend crucially on the characteristics of the source. Most of the above mentioned results are obtained for memoryless or Markovian sources, while realistic sources are often more complex. We work here with a more general source model. It includes nonmarkovian processes where the dependency on past history is unbounded. It has the advantage of making calculations possible as well. The source is associated with a so-called Variable Length Markov Chain (VLMC) (see Rissanen [12] for the seminal work, Galves-Löcherbach [7] for an overview, and [1] for a probabilistic frame). We deal with a particular VLMC source defined by an infinite comb, described hereafter.
A first browse of the literature reveals tries or suffix tries that have a height and a

saturation level both growing logarithmically with the number of words inserted. For plain tries, *i.e.* when the inserted words are independent, the results due to Pittel [11] rely on two assumptions on the source that produces the words: first, the source is uniformly mixing, second, the probability of any word decays exponentially with its length. Let us also mention the general analysis of tries by Clément *et al.* [2] for dynamical sources. For suffix tries, Szpankowski [16] obtains the same result, with a weaker mixing assumption (still uniform though) and with the same hypothesis on the measure of the words. Nevertheless, in [13], Shields states a result on prefixes of ergodic processes suggesting that the saturation level of suffix tries might not necessarily grow logarithmically with the number of words inserted.
Our aim is to exhibit two cases when the behaviour of the height or the saturation level is no longer logarithmic. The first example is the "*logarithmic comb*", for which we show that the mixing is slow in some sense, namely non uniformly polynomial (see Section 3.2 for a precise statement) and the measure of some increasing sequence of words decays polynomially. We prove in Theorem 4.8 that the height of this trie is larger than a power of n (when n is the number of inserted suffixes in the tree). The second example is the "*factorial comb*", which has a uniformly exponential mixing, thus fulfilling the mixing hypothesis of Szpankowski [16], but the measure of some increasing sequence of words decays faster than any exponential. In this case we prove in Theorem 4.9 that the saturation level is negligible with respect to $\ln n$. We prove more precisely that, almost surely, $\ell_n \in o\left(\frac{\ln n}{(\ln\ln n)^\delta}\right)$, for any $\delta > 1$.

The paper is organised as follows. In Section 2, we define a VLMC source associated with an infinite comb. In Section, 3 we give results on the mixing properties of these sources by explicitely computing the suitable generating functions in terms of the source data. In Section 4, the associated suffix tries are built, and the two uncommon behaviours are stated and shown. The methods are based on two key tools concerning successive occurence times of patterns: a duality property and the computation of generating functions. The relation between the mixing of the source and the asymptotic behaviour of the trie is highlighted by the proof of Proposition 4.7.

2 Infinite combs as sources

In this section, a VLMC probabilistic source associated with an infinite comb is defined. Moreover, we present the two examples given in introduction: the logarithmic and the factorial combs. We begin with the definition of a general

variable length Markov chain associated with a probabilized infinite comb.
The following presentation comes from [1]. Consider the binary tree (represented in Figure 1) whose finite leaves are the words $1, 01, \dots, 0^k1, \dots$ and with an infinite leaf 0^∞ as well. The set of leaves is

$$\mathcal{C} := \{0^n1,\, n \geq 0\} \cup \{0^\infty\}.$$

Each leaf is labelled with a Bernoulli distribution on $\{0, 1\}$, respectively denoted by $q_{0^k1}, k \geqslant 0$ and q_{0^∞}, *i.e.*

$$\forall k \geq 0 \quad q_{0^k1}(0) = 1 - q_{0^k1}(1).$$

This probabilized tree is called *the infinite comb*.

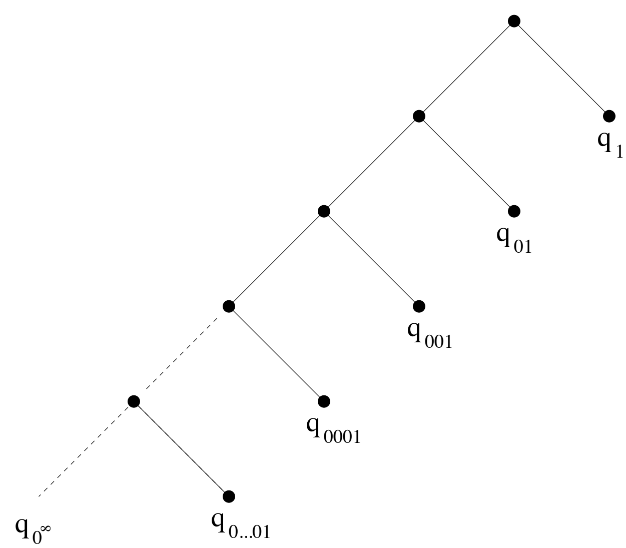

Figure 1: An infinite comb

Let $\mathcal{L} = \{0, 1\}^{-\mathbb{N}}$ be the set[5] of left-infinite words on the alphabet $\{0, 1\}$. The *prefix function*

$$\overleftarrow{\text{pref}} : \mathcal{L} = \{0, 1\}^{-\mathbb{N}} \to \mathcal{C}$$

associates to any left-infinite word (reading from right to left) its first suffix appearing as a leaf of the comb. The prefix function indicates the length of the last run of 0: for instance,

$$\overleftarrow{\text{pref}}\,(\dots 1000) = 0001 = 0^31.$$

[5]The notation $-\mathbb{N}$ stands for the set $\{n \in \mathbb{Z} \text{ such that } -n \in \mathbb{N}\}$.

The VLMC (Variable Length Markov Chain) associated with an infinite comb is the $\mathcal{L}$-valued Markov chain $(V_n)_{n\geqslant 0}$ defined by the transitions

$$\mathbb{P}(V_{n+1} = V_n\alpha | V_n) = q_{\overleftarrow{\text{pref}}\,(V_n)}(\alpha)$$

where $\alpha \in \{0, 1\}$ is a letter. Notice that the VLMC is entirely determined by the family of distributions $q_{0^\infty}, q_{0^k1}, k \geqslant 0$.
From now on, denote $c_0 = 1$ and for $n \geqslant 1$,

$$c_n := \prod_{k=0}^{n-1} q_{0^k1}(0).$$

It is proved in [1] that in the irreducible case *i.e.* when $q_{0^\infty}(0) \neq 1$, there exists a unique stationary probability measure π on $\mathcal{L}$ for $(V_n)_n$ if and only if the series $\sum c_n$ converges. From now on, we assume that this condition is fulfilled and we call

$$S(x) := \sum_{n\geqslant 0} c_n x^n \tag{1}$$

its generating function so that $S(1) = \sum_{n\geqslant 0} c_n$. For any finite word w, we denote $\pi(w) := \pi(\mathcal{L}w)$. Computations performed in [1] show that for any $n \geqslant 0$,

$$\pi(10^n) = \frac{c_n}{S(1)} \quad \text{and} \quad \pi(0^n) = \frac{\sum_{k\geqslant n} c_k}{S(1)}. \tag{2}$$

Notice that, by disjointness of events $\pi(0^n) = \pi(0^{n+1}) + \pi(10^n)$ and by stationarity, $\pi(0^n) = \pi(0^{n+1}) + \pi(0^n1)$ for all $n \geqslant 1$ so that

$$\pi(10^n) = \pi(0^n1). \tag{3}$$

If U_n denotes the final letter of V_n, the random sequence $W = U_0U_1U_2\ldots$ is a right-infinite random word. We define in this way a probabilistic source in the sense of information theory *i.e.* a mechanism that produces random words. This VLMC probabilistic source is characterized by the family of nonnegative numbers

$$p_w := \mathbb{P}(W \text{ has } w \text{ as a prefix }) = \pi(w),$$

indexed by all finite words w. This paper deals with suffix tries built from such sources. More precisely we consider two particular examples of infinite comb defined as follows.

Example 1: the logarithmic comb

The logarithmic comb is defined by $c_0 = 1$ and for $n \geqslant 1$,

$$c_n = \frac{1}{n(n+1)(n+2)(n+3)}.$$

The corresponding conditional probabilities on the leaves of the tree are

$$q_1(0) = \frac{1}{24} \quad \text{and for } n \geqslant 1, \quad q_{0^n1}(0) = 1 - \frac{4}{n+4}.$$

The expression of c_n was chosen to make the computations as simple as possible and also because the square-integrability of the waiting time of some pattern will be needed (see end of Section 4.3), guaranteed by

$$\sum_{n \geqslant 0} n^2 c_n < +\infty.$$

Example 2: the factorial comb

The conditional probabilities on the leaves are defined by

$$q_{0^n1}(0) = \frac{1}{n+2} \qquad \text{for } n \geqslant 0,$$

so that

$$c_n = \frac{1}{(n+1)!}.$$

3 Mixing properties of infinite combs

In this section, we first precise what we mean by mixing properties of a random sequence. We refer to Doukhan [4], especially for the notion of ψ-mixing defined in that book. We state in Proposition 3.2 a general result that provides the mixing coefficient for an infinite comb defined by $(c_n)_{n \geqslant 0}$ or equivalently by its generating function S. This result is then applied to our two examples. The mixing of the logarithmic comb is polynomial but not uniform, it is a very weak mixing; the mixing of the factorial comb is uniform and exponential, it is a very strong mixing. Notice that mixing properties of some infinite combs have already been investigated by Isola [10], although with a slight different language.

3.1 Mixing properties of general infinite combs

For a stationary sequence $(U_n)_{n\geqslant 0}$ with stationary measure π, we want to measure by means of a suitable coefficient the independence between two words A and B separated by n letters. The sequence is said to be "mixing" when this coefficient vanishes when n goes to $+\infty$. Among all types of mixing, we focus on one of the strongest type: ψ*-mixing*. More precisely, for $0 \leqslant m \leqslant +\infty$, denote by $\mathcal{F}_0^m$ the σ-algebra generated by $\{U_k, 0 \leqslant k \leqslant m\}$ and introduce for $A \in \mathcal{F}_0^m$ and $B \in \mathcal{F}_0^\infty$ the mixing coefficient

$$\begin{aligned}\psi(n, A, B) &:= \frac{\pi(A \cap T^{-(m+1)-n}B) - \pi(A)\pi(B)}{\pi(A)\pi(B)} \\ &= \frac{\sum_{|w|=n} \pi(AwB) - \pi(A)\pi(B)}{\pi(A)\pi(B)}, \qquad (4)\end{aligned}$$

where T is the shift map and where the sum runs over the finite words w with length $|w| = n$.
A sequence $(U_n)_{n\geqslant 0}$ is called ψ*-mixing* whenever

$$\lim_{n\to\infty} \sup_{m\geqslant 0, A\in\mathcal{F}_0^m, B\in\mathcal{F}_0^\infty} |\psi(n, A, B)| = 0.$$

In this definition, the convergence to zero is uniform over all words A and B. This is not going to be the case in our first example. As in Isola [10], we widely use the renewal properties of infinite combs (see Lemma 3.1) but more detailed results are needed, in particular we investigate the lack of uniformity for the logarithmic comb.

Notations and Generating functions

• For a comb, recall that S is the generating function of the nonincreasing sequence $(c_n)_{n\geqslant 0}$ defined by (1).
• Set $\rho_0 = 0$ and for $n \geqslant 1$,

$$\rho_n := c_{n-1} - c_n,$$

with generating function

$$P(x) := \sum_{n\geqslant 1} \rho_n x^n.$$

• Define the sequence $(u_n)_{n\geqslant 0}$ by $u_0 = 1$ and for $n \geqslant 1$,

$$u_n := \frac{\pi(U_0 = 1, U_n = 1)}{\pi(1)} = \frac{1}{\pi(1)} \sum_{|w|=n-1} \pi(1w1), \qquad (5)$$

and let

$$U(x) := \sum_{n \geqslant 0} u_n x^n$$

denote its generating function. Hereunder is stated a key lemma that will be widely used in Proposition 3.2. In some sense, this kind of relation (sometimes called Renewal Equation) reflects the renewal properties of the infinite comb.

Lemma 3.1 *The sequences $(u_n)_{n \geqslant 0}$ and $(\rho_n)_{n \geqslant 0}$ are connected by the relations:*

$$\forall n \geqslant 1, \quad u_n = \rho_n + u_1 \rho_{n-1} + \ldots + u_{n-1} \rho_1$$

and (consequently)

$$U(x) = \sum_{n=0}^{\infty} u_n x^n = \frac{1}{1 - P(x)} = \frac{1}{(1-x)S(x)}.$$

Proof. For a finite word $w = \alpha_1 \ldots \alpha_m$ such that $w \neq 0^m$, let $l(w)$ denote the position of the last 1 in w, that is $l(w) := \max\{1 \leq i \leq m,\ \alpha_i = 1\}$. Then, the sum in the expression (5) of u_n can be decomposed as follows:

$$\sum_{|w|=n-1} \pi(1w1) = \pi(10^{n-1}1) + \sum_{i=1}^{n-1} \sum_{\substack{|w|=n-1 \\ l(w)=i}} \pi(1w1).$$

Now, by disjoint union $\pi(10^{n-1}) = \pi(10^{n-1}1) + \pi(10^n)$, so that

$$\pi(10^{n-1}1) = \pi(1)(c_{n-1} - c_n) = \pi(1)\rho_n.$$

In the same way, for $w = \alpha_1 \ldots \alpha_{n-1}$, if $l(w) = i$ then $\pi(1w1) = \pi(1\alpha_1 \ldots \alpha_{i-1}1)\rho_{n-i}$, so that

$$\begin{aligned} u_n &= \rho_n + \sum_{i=1}^{n-1} \rho_{n-i} \frac{1}{\pi(1)} \sum_{|w|=i-1} \pi(1w1) \\ &= \rho_n + \sum_{i=1}^{n-1} \rho_{n-i} u_i, \end{aligned}$$

which leads to $U(x) = (1 - P(x))^{-1}$ by summation. $\square$

Mixing coefficients

The mixing coefficients $\psi(n,A,B)$ are expressed as the n-th coefficient in the series expansion of an analytic function $M^{A,B}$ which is given in terms of S and U. The notation $[x^n]A(x)$ means the coefficient of x^n in the power expansion of $A(x)$ at the origin. Denote the remainders associated with the series $S(x)$ by

$$r_n := \sum_{k\geqslant n} c_k, \quad R_n(x) := \sum_{k\geqslant n} c_k x^k$$

and for $a\geqslant 0$, define the "shifted" generating function

$$P_a(x) := \frac{1}{c_a}\sum_{n\geqslant 1}\rho_{a+n}x^n = x + \frac{x-1}{c_a x^a}R_{a+1}(x). \tag{6}$$

Proposition 3.2 *For any finite word A and any word B, the identity*

$$\psi(n,A,B) = [x^{n+1}]M^{A,B}(x)$$

holds for the generating functions $M^{A,B}$ respectively defined by:

i) *if $A=A'1$ and $B=1B'$ where A' and B' are any finite words, then*

$$M^{A,B}(x) = M(x) := \frac{S(x)-S(1)}{(x-1)S(x)};$$

ii) *if $A=A'10^a$ and $B=0^b1B'$ where A' and B' are any finite words and $a+b\geqslant 1$, then*

$$M^{A,B}(x) := S(1)\frac{c_{a+b}}{c_a c_b}P_{a+b}(x) + U(x)\left[S(1)P_a(x)P_b(x) - S(x)\right];$$

iii) *if $A=0^a$ and $B=0^b$ with $a,b\geqslant 1$, then*

$$M^{A,B}(x) := S(1)\frac{1}{r_a r_b}\sum_{n\geqslant 1} r_{a+b+n}x^n + U(x)\left[\frac{S(1)R_a(x)R_b(x)}{r_a r_b x^{a+b-2}} - S(x)\right];$$

iv) *if $A=A'10^a$ and $B=0^b$ where A' is any finite words and $a,b\geqslant 0$, then*

$$M^{A,B}(x) := S(1)\frac{1}{c_a r_b x^{a+b-1}}R_{a+b}(x) + U(x)\left[\frac{S(1)P_a(x)R_b(x)}{c_a r_b x^{b-1}} - S(x)\right];$$

v) *if* $A = 0^a$ *and* $B = 0^b1B'$ *where* B' *is any finite words and* $a, b \geqslant 0$*, then*

$$M^{A,B}(x) := S(1)\frac{1}{r_a c_b x^{a+b-1}} R_{a+b}(x) + U(x)\left[\frac{S(1)R_a(x)P_b(x)}{r_a c_b x^{a-1}} - S(x)\right].$$

Remark 3.3 *It is worth noticing that the asymptotics of* $\psi(n, A, B)$ *may not be uniform in all words* A *and* B*. We call this kind of system non-uniformly* ψ*-mixing. It may happen that* $\psi(n, A, B)$ *goes to zero for any fixed* A *and* B*, but (for example, in case* ***iii)****) the larger* a *or* b*, the slower the convergence, preventing it from being uniform.*

Proof. The following identity has been established in [1] (see formula (17) in that paper) and will be used many times in the sequel. For any two finite words w and w',

$$\pi(w1w')\pi(1) = \pi(w1)\pi(1w'). \tag{7}$$

i) If $A = A'1$ and $B = 1B'$, then (7) yields

$$\pi(AwB) = \pi(A'1w1B') = \frac{\pi(A'1)}{\pi(1)}\pi(1w1B') = S(1)\pi(A)\pi(B)\frac{\pi(1w1)}{\pi(1)}.$$

So

$$\psi(n, A, B) = S(1)u_{n+1} - 1$$

and by Lemma 3.1, the result follows.

ii) Let $A = A'10^a$ and $B = 0^b1B'$ with $a, b \geqslant 0$ and $a + b \neq 0$. To begin with,

$$\pi(AwB) = \frac{1}{\pi(1)}\pi(A'1)\pi(10^a w 0^b 1B') = \frac{1}{\pi(1)^2}\pi(A'1)\pi(10^a w 0^b 1)\pi(1B').$$

Furthermore, $\pi(A) = c_a\pi(A'1)$ and by (3), $\pi(0^b1) = \pi(10^b)$, so it comes

$$\pi(B) = \frac{1}{\pi(1)}\pi(0^b1)\pi(1B') = \frac{\pi(10^b)}{\pi(1)}\pi(1B') = c_b\pi(1B').$$

Therefore,

$$\pi(AwB) = \frac{\pi(A)\pi(B)}{c_a c_b \pi(1)^2}\pi(10^a w 0^b 1).$$

Using $\pi(1)S(1) = 1$, this proves

$$\psi(n, A, B) = S(1)\frac{v_n^{a,b}}{c_a c_b} - 1$$

where

$$v_n^{a,b} := \frac{1}{\pi(1)} \sum_{|w|=n} \pi(10^a w 0^b 1).$$

As in the proof of the previous lemma, if $w = \alpha_1 \ldots \alpha_m$ is any finite word different from 0^m, we call $f(w) := \min\{1 \le i \le m, \alpha_i = 1\}$ the first place where 1 can be seen in w and recall that $l(w)$ denotes the last place where 1 can be seen in w. One has

$$\sum_{|w|=n} \pi(10^a w 0^b 1) = \pi(10^{a+n+b}1) + \sum_{1\le i\le j\le n} \sum_{\substack{|w|=n \\ f(w)=i, l(w)=j}} \pi(10^a w 0^b 1).$$

If $i = j$ then w is the word $0^{i-1}10^{n-i}$, else w is of the form $0^{i-1}1w'10^{n-j}$, with $|w'| = j - i - 1$. Hence, the previous sum can be rewritten as

$$\begin{aligned}\sum_{|w|=n} \pi(10^a w 0^b 1) &= \pi(1)\rho_{a+b+n+1} + \pi(1)\sum_{i=1}^{n} \rho_{a+i}\rho_{n+1-i+b} \\ &\quad + \sum_{1\leqslant i<j\leqslant n} \sum_{\substack{w \\ |w|=j-i-1}} \pi(10^{a+i-1}1w10^{n-j+b}1).\end{aligned}$$

Equation (7) shows

$$\begin{aligned}\pi(10^{a+i-1}1w10^{n-j+b}1) &= \frac{\pi(10^{a+i-1}1)}{\pi(1)} \frac{\pi(1w1)}{\pi(1)} \pi(10^{n-j+b}1) \\ &= \rho_{a+i}\rho_{n+1-j+b}\pi(1w1).\end{aligned}$$

This implies:

$$v_n^{a,b} = \rho_{a+b+n+1} + \sum_{i=1}^{n} \rho_{a+i}\rho_{n+1-i+b} + \sum_{1\le i<j\le n} \rho_{a+i}\rho_{n+1-j+b} \sum_{w, |w|=j-i-1} \frac{\pi(1w1)}{\pi(1)}.$$

Recalling that $u_0 = 1$, one gets

$$v_n^{a,b} = \rho_{a+b+n+1} + \sum_{1\le i\le j\le n} \rho_{a+i}\rho_{n+1-j+b} u_{j-i}$$

which gives the result ***ii)*** with Lemma 3.1.

iii) Let $A = 0^a$ and $B = 0^b$ with $a, b \geqslant 1$. Set

$$v_n^{a,b} := \frac{1}{\pi(1)} \sum_{|w|=n} \pi(0^a w 0^b).$$

First, recall that, due to (2), $\pi(A) = \pi(1)r_a$ and $\pi(B) = \pi(1)r_b$. Consequently,

$$\psi(n, A, B) = \frac{\pi(1)v_n^{a,b} - \pi(A)\pi(B)}{\pi(A)\pi(B)} = S(1)\frac{v_n^{a,b}}{r_a r_b} - 1.$$

Let w be a finite word with $|w| = n$. If $w = 0^n$, then

$$\pi(AwB) = \pi(0^{a+n+b}) = \pi(1)r_{a+b+n}.$$

If not, let $f(w)$ denote as before the first position of 1 in w and $l(w)$ the last one in w. If $f(w) = l(w)$, then

$$\begin{aligned} \pi(AwB) &= \pi(0^{a+f(w)-1}10^{n-f(w)+b}) \\ &= \tfrac{1}{\pi(1)}\pi(0^{a+f(w)-1}1)\pi(10^{n-f(w)+b}) = \pi(1)c_{a+f(w)-1}c_{n-f(w)+b}. \end{aligned}$$

If $f(w) < l(w)$, then writing $w = w_1 \dots w_n$,

$$\begin{aligned} \pi(AwB) &= \pi(0^{a+f(w)-1}1w_{f(w)+1} \dots w_{l(w)-1}10^{n-l(w)}) \\ &= \tfrac{1}{\pi(1)^2}\pi(0^{a+f(w)-1}1)\pi(1w_{f(w)+1} \dots w_{l(w)-1}1)\pi(10^{n-l(w)+b}). \end{aligned}$$

Summing yields

$$\begin{aligned} v_n^{a,b} &= r_{a+b+n} + \sum_{i=1}^{n} c_{a+i-1}c_{n+b-i} + \sum_{\substack{i,j=1 \\ i<j}}^{n} \sum_{\substack{w, \\ |w|=j-i-1}} c_{a+i-1}\frac{\pi(1w1)}{\pi(1)}c_{n+b-j} \\ &= r_{a+b+n} + \sum_{1 \le i \le j \le n} c_{a+i-1}c_{n+b-j}u_{j-i}, \end{aligned}$$

which gives the desired result. The last two items, left to the reader, follow the same guidelines. □

3.2 Mixing of the logarithmic infinite comb

Consider the first example in Section 2, that is the probabilized infinite comb defined by $c_0 = 1$ and for any $n \geqslant 1$ by

$$c_n = \frac{1}{n(n+1)(n+2)(n+3)}.$$

When $|x| < 1$, the series $S(x)$ writes as follows

$$S(x) = \frac{47}{36} - \frac{5}{12x} + \frac{1}{6x^2} + \frac{(1-x)^3\log(1-x)}{6x^3} \tag{8}$$

and

$$S(1)=\frac{19}{18}.$$

With Proposition 3.2, the asymptotics of the mixing coefficient comes from singularity analysis of the generating functions $M^{A,B}$.

Proposition 3.4 *The VLMC defined by the logarithmic infinite comb has a non-uniform polynomial mixing of the following form: for any finite words A and B, there exists a positive constant $C_{A,B}$ such that for any $n\geqslant 1$,*

$$|\psi(n,A,B)|\leqslant\frac{C_{A,B}}{n^3}.$$

Remark 3.5 *The $C_{A,B}$ cannot be bounded above by some constant that does not depend on A and B, as can be seen hereunder in the proof. Indeed, we show that if a and b are positive integers,*

$$\psi(n,0^a,0^b)\sim\frac{1}{3}\left(\frac{S(1)}{r_ar_b}-\frac{1}{r_a}-\frac{1}{r_b}+\frac{1}{S(1)}\right)\frac{1}{n^3}$$

as n goes to infinity. In particular, $\psi(n,0,0^n)$ tends to the positive constant $\frac{13}{6}$.

Proof of Proposition 3.4.
For any finite words A and B in case ***i)*** of Proposition 3.2, one deals with $U(x)=\left((1-x)S(x)\right)^{-1}$ which has 1 as a unique dominant singularity. Indeed, 1 is the unique dominant singularity of S, so that the dominant singularities of U are 1 or zeroes of S contained in the closed unit disc. But S does not vanish on the closed unit disc, because for any z such that $|z|\leqslant 1$,

$$|S(z)|\geqslant 1-\sum_{n\geqslant 1}\frac{1}{n(n+1)(n+2)(n+3)}=1-(S(1)-1)=\frac{17}{18}.$$

Since

$$M(x)=\frac{S(x)-S(1)}{(x-1)S(x)}=S(1)U(x)-\frac{1}{1-x},$$

the unique dominant singularity of M is 1, and when x tends to 1 in the unit disc, (8) leads to

$$M(x)=A(x)-\frac{1}{6S(1)}(1-x)^2\log(1-x)+\mathcal{O}\big((1-x)^3\log(1-x)\big)$$

where $A(x)$ is a polynomial of degree 2. Using a classical transfer theorem based on the analysis of the singularities of M (see Flajolet and Sedgewick [6, section VI.4, p. 393 and special case p. 387]), we get

$$\psi(n-1, w1, 1w') = [x^n]M(x) = \frac{1}{3S(1)}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right).$$

The cases ***ii)***, ***iii)***, ***iv)*** and ***v)*** of Proposition 3.2 are of the same kind, and we completely deal with case ***iii)***.

Case ***iii)***: words of the form $A = 0^a$ and $B = 0^b$, $a, b \geqslant 1$. As shown in Proposition 3.2, one has to compute the asymptotics of the n-th coefficient of the Taylor series of the function

$$M^{a,b}(x) := S(1)\frac{1}{r_a r_b}\sum_{n\geqslant 1} r_{a+b+n}x^n + U(x)\left[\frac{S(1)R_a(x)R_b(x)}{r_a r_b x^{a+b-2}} - S(x)\right]. \tag{9}$$

The contribution of the left-hand term of this sum is directly given by the asymptotics of the remainder

$$r_n = \sum_{k\geqslant n} c_k = \frac{1}{3n(n+1)(n+2)} = \frac{1}{3n^3} + \mathcal{O}\left(\frac{1}{n^4}\right).$$

By means of singularity analysis, we deal with the right-hand term

$$N^{a,b}(x) := U(x)\left[\frac{S(1)R_a(x)R_b(x)}{r_a r_b x^{a+b-2}} - S(x)\right].$$

Since 1 is the only dominant singularity of S and U and consequently of any R_a, it suffices to compute an expansion of $N^{a,b}(x)$ at $x = 1$. It follows from (8) that U, S and R_a admit expansions near 1 of the forms

$$U(x) = \frac{1}{S(1)(1-x)} + \text{polynomial} + \frac{1}{6S(1)^2}(1-x)^2\log(1-x) + \mathcal{O}(1-x)^2,$$
$$S(x) = \text{polynomial} + \frac{1}{6}(1-x)^3\log(1-x) + \mathcal{O}(1-x)^3,$$

and

$$R_a(x) = \text{polynomial} + \frac{1}{6}(1-x)^3\log(1-x) + \mathcal{O}(1-x)^3.$$

Consequently,

$$N^{a,b}(x) = \frac{1}{6}\left(\frac{1}{r_a} + \frac{1}{r_b} - \frac{1}{S(1)}\right)(1-x)^2\log(1-x) + \mathcal{O}(1-x)^2$$

in a neighbourhood of 1 in the unit disc so that, by singularity analysis,

$$[x^n]N^{a,b}(x) = -\frac{1}{3}\left(\frac{1}{r_a}+\frac{1}{r_b}-\frac{1}{S(1)}\right)\frac{1}{n^3}+o\left(\frac{1}{n^3}\right).$$

Consequently (9) leads to

$$\psi(n-1,0^a,0^b) = [x^n]M^{a,b}(x) \sim \frac{1}{3}\left(\frac{S(1)}{r_ar_b}-\frac{1}{r_a}-\frac{1}{r_b}+\frac{1}{S(1)}\right)\frac{1}{n^3}$$

as n tends to infinity, showing the mixing inequality and the non uniformity. The remaining cases ***ii)***, ***iv)*** and ***v)*** are of the same flavour. □

3.3 Mixing of the factorial infinite comb

Consider now the second Example in Section 2, that is the probabilized infinite comb defined by

$$\forall n \in \mathbb{N},\ c_n = \frac{1}{(n+1)!}.$$

With previous notations, one gets

$$S(x) = \frac{e^x-1}{x} \text{ and } U(x) = \frac{x}{(1-x)(e^x-1)}.$$

Proposition 3.6 *The VLMC defined by the factorial infinite comb has a uniform exponential mixing of the following form: there exists a positive constant C such that for any $n \geqslant 1$ and for any finite words A and B,*

$$|\psi(n,A,B)| \leqslant \frac{C}{(2\pi)^n}.$$

Proof.

i) First case of mixing in Proposition 3.2: $A = A'1$ and $B = 1B'$.

Because of Proposition 3.2, the proof consists in computing the asymptotics of $[x^n]M(x)$. We make use of singularity analysis. The dominant singularities of

$$M(x) = \frac{S(x)-S(1)}{(x-1)S(x)}$$

are readily seen to be $2i\pi$ and $-2i\pi$, and

$$M(x) \underset{2i\pi}{\sim} \frac{1-e}{1-2i\pi}\cdot\frac{1}{1-\frac{z}{2i\pi}}.$$

The behaviour of M in a neighbourhood of $-2i\pi$ is obtained by complex conjugacy. Singularity analysis via transfer theorem provides thus that

$$[x^n]M(x) \underset{n\to+\infty}{\sim} \frac{2(e-1)}{1+4\pi^2}\left(\frac{1}{2\pi}\right)^n \epsilon_n$$

where

$$\epsilon_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2\pi & \text{if } n \text{ is odd.} \end{cases}$$

ii) Second case of mixing: $A = A'10^a$ and $B = 0^b1B'$.

Because of Proposition 3.2, one has to compute $[x^n]M^{a,b}(x)$ with

$$M^{a,b}(x) := S(1)\frac{c_{a+b}}{c_a c_b}P_{a+b}(x) + \frac{1}{S(x)}\cdot\frac{1}{1-x}\Big[S(1)P_a(x)P_b(x) - S(x)\Big],$$

where P_{a+b} is an entire function. In this last formula, the brackets contain an entire function that vanishes at 1 so that the dominant singularities of $M^{a,b}$ are again those of S^{-1}, namely $\pm 2i\pi$. The expansion of $M^{a,b}(x)$ at $2i\pi$ writes thus

$$M^{a,b}(x) \underset{2i\pi}{\sim} \frac{-S(1)P_a(2i\pi)P_b(2i\pi)}{1-2i\pi}\cdot\frac{1}{1-\frac{x}{2i\pi}}$$

which implies, by singularity analysis, that

$$[x^n]M^{a,b}(x) \underset{n\to+\infty}{\sim} 2\Re\left(\frac{1-e}{1-2i\pi}\cdot\frac{P_a(2i\pi)P_b(2i\pi)}{(2i\pi)^n}\right).$$

Besides, the remainder of the exponential series satisfies

$$\sum_{n\geqslant a}\frac{x^n}{n!} = \frac{x^a}{a!}\left(1+\frac{x}{a}+\mathcal{O}(\frac{1}{a})\right) \tag{10}$$

when a tends to infinity. Consequently, by Formula (6), $P_a(2i\pi)$ tends to $2i\pi$ as a tends to infinity so that one gets a positive constant C_1 that does not depend on a and b such that for any $n \geqslant 1$,

$$|\psi(n,A,B)| \leqslant \frac{C_1}{(2\pi)^n}.$$

iii) Third case of mixing: $A = 0^a$ and $B = 0^b$.

This time, one has to compute $[x^n]M^{a,b}(x)$ with

$$M^{a,b}(x) := S(1)\frac{1}{r_a r_b}\sum_{n\geqslant 1} r_{a+b+n}x^n + U(x)\left[\frac{S(1)R_a(x)R_b(x)}{r_a r_b x^{a+b-2}} - S(x)\right]$$

the first term being an entire function. Here again, the dominant singularities of $M^{a,b}$ are located at $\pm 2i\pi$ and

$$M^{a,b}(x) \underset{2i\pi}{\sim} \frac{-S(1)R_a(2i\pi)R_b(2i\pi)}{(1-2i\pi)r_a r_b(2i\pi)^{a+b-2}} \cdot \frac{1}{1-\frac{x}{2i\pi}}$$

which implies, by singularity analysis, that

$$\psi(n-1, A, B) \underset{n\to+\infty}{\sim} 2\Re\left(\frac{1-e}{1-2i\pi} \cdot \frac{R_a(2i\pi)R_b(2i\pi)}{r_a r_b(2i\pi)^{a+b-2}}\frac{1}{(2i\pi)^n}\right).$$

Once more, because of (10), this implies that there is a positive constant C_2 independent of a and b and such that for any $n \geqslant 1$,

$$|\psi(n, A, B)| \leqslant \frac{C_2}{(2\pi)^n}.$$

iv) and ***v)****: both remaining cases of mixing that respectively correspond to words of the form $A = A'10^a$, $B = 0^b$ and $A = 0^a$, $B = 0^b1B'$ are of the same vein and lead to similar results.* □

4 Height and saturation level of suffix tries

In this section, we consider a suffix trie process $(\mathcal{T}_n)_{n\geq 1}$ associated with an infinite random word generated by an infinite comb. A precise definition of tries and suffix tries is given in section 4.1. We are interested in the height and the saturation level of such a suffix trie.

Our method to study these two parameters uses a duality property à la Pittel developed in Section 4.2, together with a careful and explicit calculation of the generating function of the second occurrence of a word (in Section 4.3) which can be achieved for any infinite comb. These calculations are not so intricate because they are strongly related to the mixing coefficient and the mixing properties detailed in Section 3.

More specifically, we look at our two favourite examples, the logarithmic comb and the factorial comb. We prove in Section 4.5 that the height of the first one is not logarithmic but polynomial and in Section 4.6 that the saturation level of the second one is not logarithmic either but negligibly smaller. Remark that despite the very particular form of the comb in the wide family of variable length Markov models, the comb sources provide a spectrum of asymptotic behaviours for the suffix tries.

4.1 Suffix tries

Let $(y_n)_{n\geqslant 1}$ be a sequence of right-infinite words on $\{0,1\}$. With this sequence is associated a **trie process** $(\mathcal{T}_n)_{n\geq 1}$ which is a planar unary-binary tree increasing sequence defined the following way. The tree $\mathcal{T}_n$ contains the words $y_1,\dots,y_n$ in its leaves. It is obtained by a sequential construction, inserting the words y_n successively. At the beginning, $\mathcal{T}_1$ is the tree containing the root and the leaf $0\dots$ (resp. the leaf $1\dots$) if y_1 begins with 0 (resp. with 1). For $n\geq 2$, given the tree $\mathcal{T}_{n-1}$, the word y_n is inserted as follows. We go through the tree along the branch whose nodes are encoded by the successive prefixes of y_n; when the branch ends, if an internal node is reached, then y_n is inserted at the free leaf, else we make the branch grow comparing the next letters of both words until they can be inserted in two different leaves. As one can clearly see on Figure 2 a trie is not a complete tree and the insertion of a word can make a branch grow by more than one level. Notice that if the trie contains a finite word w as internal node, there are at least two already inserted infinite words y_n that have w as a prefix. This indicates why the *second* occurrence of a word is prominent in the growing of a trie.

Let $m := a_1a_2a_3\dots$ be an infinite word on $\{0,1\}$. The **suffix trie** $\mathcal{T}_n$ (with n leaves) associated with m is the trie built from the first n suffixes of m one obtains by deleting successively the left-most letter, that is

$$y_1 = m,\ y_2 = a_2a_3a_4\dots,\ y_3 = a_3a_4\dots,\ \dots,\ y_n = a_na_{n+1}\dots$$

For a given trie $\mathcal{T}_n$, we are mainly interested in the *height* H_n which is the maximal depth of an internal node of $\mathcal{T}_n$ and the *saturation level* ℓ_n which is the maximal depth up to which all the internal nodes are present in $\mathcal{T}_n$. Formally, if $\partial\mathcal{T}_n$ denotes the set of leaves of $\mathcal{T}_n$,

$$\begin{aligned} H_n &= \max_{u\in\mathcal{T}_n\setminus\partial\mathcal{T}_n}\big\{|u|\big\} \\ \ell_n &= \max\big\{j\in\mathbb{N}\,|\ \#\{u\in\mathcal{T}_n\setminus\partial\mathcal{T}_n, |u|=j\}=2^j\big\}. \end{aligned}$$

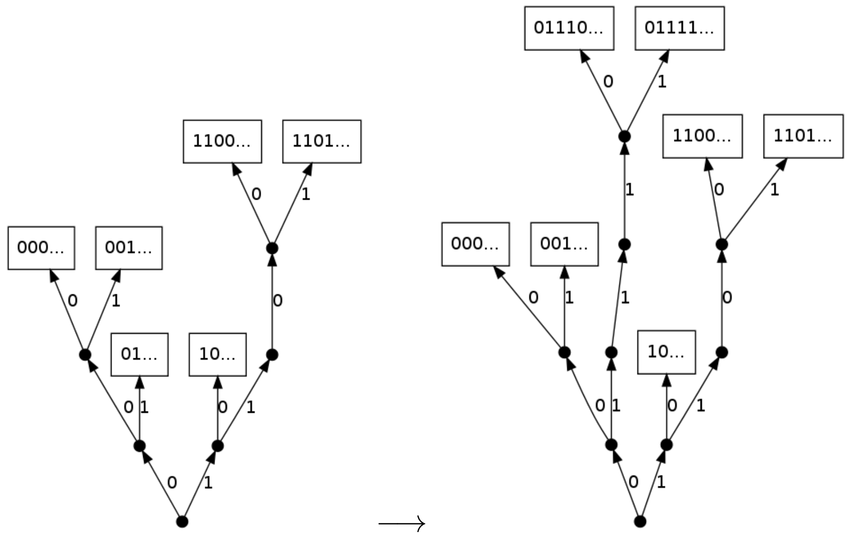

Figure 2: Last steps of the construction of a trie built from the set $(000\ldots, 10\ldots, 1101\ldots, 001,\ldots, 01110\ldots, 1100\ldots, 01111\ldots)$.

See Figure 3 for an example. Note that the saturation level should not be mistaken for the shortest path up to a leaf.

4.2 Duality

Let $(U_n)_{n\geqslant 1}$ be an infinite random word generated by some infinite comb and $(\mathcal{T}_n)_{n\geqslant 1}$ be the associated suffix trie process. We denote by $\mathcal{R}$ the set of right-infinite words. Besides, we define hereunder two random variables having a key role in the proof of Theorem 4.8 and Theorem 4.9. This method goes back to Pittel [11].
Let $s \in \mathcal{R}$ be a deterministic infinite sequence and $s^{(k)}$ its prefix of length k. For $n \geq 1$,

$$\begin{aligned} X_n(s) &:= \begin{cases} 0 \text{ if } s^{(1)} \text{ is not in } \mathcal{T}_n \\ \max\{k \geqslant 1 \mid \text{the word } s^{(k)} \text{ is already in } \mathcal{T}_n \setminus \partial\mathcal{T}_n\}, \end{cases} \\ T_k(s) &:= \min\{n \geqslant 1 \mid X_n(s) = k\}, \end{aligned}$$

where "$s^{(k)}$ is in $\mathcal{T}_n \setminus \partial\mathcal{T}_n$" stands for: there exists an internal node v in $\mathcal{T}_n$ such that $s^{(k)}$ encodes v. For any $k \geqslant 1$, $T_k(s)$ denotes the number of leaves of the first tree "containing" $s^{(k)}$. See Figure 4 for an example. Thus, the saturation level

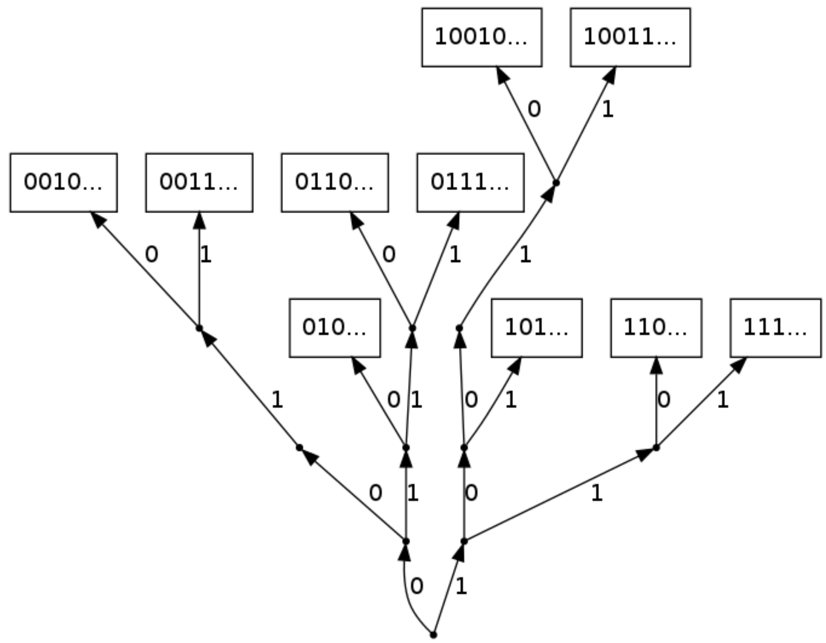

Figure 3: Suffix trie $\mathcal{T}_{10}$ associated with the word $1001011001110\ldots$. Here, $H_{10} = 4$ and $\ell_{10} = 2$.

ℓ_n and the height H_n can be described using $X_n(s)$:

$$\ell_n = \min_{s\in\mathcal{R}} X_n(s) \quad \text{and} \quad H_n = \max_{s\in\mathcal{R}} X_n(s). \tag{11}$$

Notice that the notation $X_n(s)$ is not exactly the same as in Pittel [11] so that ℓ_n does not correspond to the shortest path in $\mathcal{T}_n$.
Remark that $X_n(s)$ and $T_k(s)$ are in duality in the following sense: for all positive integers k and n, one has the equality of the events

$$\{X_n(s) \geqslant k\} = \{T_k(s) \leqslant n\}. \tag{12}$$

The random variable $T_k(s)$ (if $k \geqslant 2$) also represents the waiting time of the second occurrence of the deterministic word $s^{(k)}$ in the random sequence $(U_n)_{n\geq 1}$, *i.e.* one has to wait $T_k(s)$ for the source to create a prefix containing exactly two occurrences of $s^{(k)}$. More precisely, for $k \geqslant 2$, $T_k(s)$ can be rewritten as

$$T_k(s) = \min\Big\{n \geqslant 1 \;\Big| U_n U_{n+1}\ldots U_{n+k-1} = s^{(k)} \text{ and } \exists! j < n \text{ such that } U_j U_{j+1}\ldots U_{j+k-1} = s^{(k)}\Big\}.$$

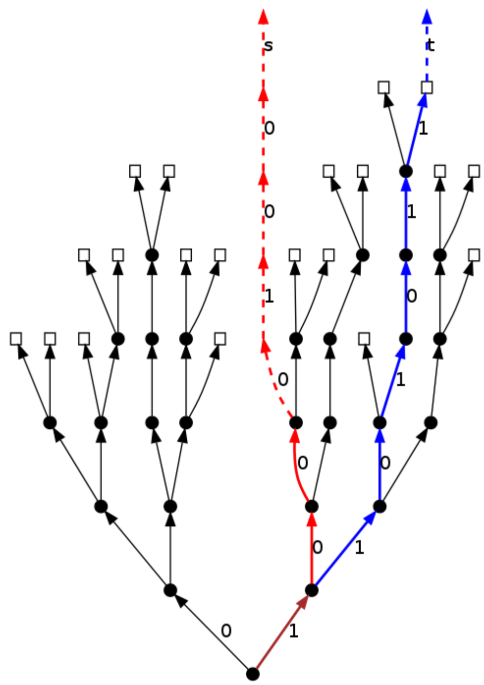

Figure 4: Example of suffix trie with $n = 20$ words. The saturation level is reached for any sequence having 1000 as prefix (in red); $\ell_{20} = X_{20}(s) = 3$ and thus $T_3(s) \leqslant 20$. The height (related to the maximum of X_{20}) is realized for any sequence of the form $110101\ldots$ (in blue) and $H_{20} = 6$. Remark that the shortest branch has length 4 whereas the saturation level ℓ_n is equal to 3.

Notice that $T_k(s)$ denotes the *beginning* of the second occurrence of $s^{(k)}$ whereas in [1], $\tau^{(2)}\left(s^{(k)}\right)$ denotes the *end* of the second occurrence of $s^{(k)}$, so that

$$\tau^{(2)}\left(s^{(k)}\right) = T_k(s) + k. \tag{13}$$

More generally, in [1] , for any $r \geqslant 1$, the random waiting time $\tau^{(r)}(w)$ is defined as the end of the r-th occurrence of w in the sequence $(U_n)_{n\geqslant 1}$ and the generating function of the $\tau^{(r)}$ is calculated. We go over these calculations in the sequel.

4.3 Successive occurence times generating functions

Proposition 4.7 *Let $k \geqslant 1$. Let also $w = 10^{k-1}$ and $\tau^{(2)}(w)$ be the end of the second occurrence of w in a sequence generated by a comb defined by $(c_n)_{n\geqslant 0}$. Let S and U be the ordinary generating functions defined in Section 3.1. The* *probability generating function of* $\tau^{(2)}(w)$ *is*

$$\Phi_w^{(2)}(x) = \frac{c_{k-1}^2 x^{2k-1}\big(U(x)-1\big)}{S(1)(1-x)\big[1+c_{k-1}x^{k-1}(U(x)-1)\big]^2}.$$

Furthermore, as soon as $\sum_{n\geqslant 1} n^2 c_n < \infty$*, the random variable* $\tau^{(2)}(w)$ *is square-integrable and*

$$\mathbb{E}(\tau^{(2)}(w)) = \frac{2S(1)}{c_{k-1}} + o\left(\frac{1}{c_{k-1}}\right), \quad \mathrm{Var}(\tau^{(2)}(w)) = \frac{2S(1)^2}{c_{k-1}^2} + o\left(\frac{1}{c_{k-1}^2}\right). \tag{14}$$

Proof. For any $r \geqslant 1$, let $\tau^{(r)}(w)$ denote the end of the r-th occurrence of w in a random sequence generated by a comb and $\Phi_w^{(r)}$ its probability generating function. The reversed word of $c = \alpha_1 \ldots \alpha_N$ will be denoted by the overline $\overline{c} := \alpha_N \ldots \alpha_1$
We use a result of [1] that computes these generating functions in terms of stationary probabilities $q_c^{(n)}$. These probabilities measure the occurrence of a finite word after n steps, conditioned to start from the word $\overline{c}$. More precisely, for any finite words u and $\overline{c}$ and for any $n \geqslant 0$, let

$$q_c^{(n)}(u) := \pi\left(U_{n-|u|+|c|+1}\ldots U_{n+|c|} = u \middle| U_1 \ldots U_{|c|} = \overline{c}\right).$$

It is shown in [1] that, for $|x| < 1$,

$$\Phi_w^{(1)}(x) = \frac{x^k \pi(w)}{(1-x)S_w(x)}$$

and for $r \geq 1$,

$$\Phi_w^{(r)}(x) = \Phi_w^{(1)}(x)\left(1 - \frac{1}{S_w(x)}\right)^{r-1}$$

where

$$\begin{aligned} S_w(x) &:= C_w(x) + \sum_{n=k}^{\infty} q^{(n)}_{\overleftarrow{\mathrm{pref}}\,(w)}(w)x^n, \\ C_w(x) &:= 1 + \sum_{j=1}^{k-1} \mathbf{1}_{\{w_{j+1}\ldots w_k = w_1 \ldots w_{k-j}\}} q^{(j)}_{\overleftarrow{\mathrm{pref}}\,(w)}\left(w_{k-j+1}\ldots w_k\right) x^j. \end{aligned}$$

In the particular case when $w = 10^{k-1}$, then $\overleftarrow{\mathrm{pref}}\,(w) = \overline{w} = 0^{k-1}1$ and $\pi(w) = \frac{c_{k-1}}{S(1)}$. Moreover, Definition (4) of the mixing coefficient and Proposition 3.2 ***i)*** imply successively that

$$\begin{aligned}
q^{(n)}_{\overleftarrow{\mathrm{pref}}\,(w)}(w) &= \pi\Big(U_{n+k-|w|+1}\dots U_{n+k} = w\Big|U_1\dots U_k = \overleftarrow{\mathrm{pref}}\,(w)\Big)\\
&= \pi(w)\Big(\psi\big(n-k, \overleftarrow{\mathrm{pref}}\,(w), w\big) + 1\Big)\\
&= \pi(w)S(1)u_{n-k+1}\\
&= c_{k-1}u_{n-k+1},
\end{aligned}$$

This relation makes more explicit the link between successive occurence times and mixing. This leads to

$$\sum_{n\geqslant k} q^{(n)}_{\overleftarrow{\mathrm{pref}}\,(w)}(w)x^n = c_{k-1}x^{k-1}\sum_{n\geqslant 1} u_n x^n = c_{k-1}x^{k-1}\big(U(x)-1\big).$$

Furthermore, there is no auto-correlation structure inside w so that $C_w(x) = 1$ and

$$S_w(x) = 1 + c_{k-1}x^{k-1}\big(U(x)-1\big).$$

This entails

$$\Phi^{(1)}_w(x) = \frac{c_{k-1}x^k}{S(1)(1-x)\left[1 + c_{k-1}x^{k-1}\big(U(x)-1\big)\right]}$$

and

$$\begin{aligned}
\Phi^{(2)}_w(x) &= \Phi^{(1)}_w(x)\left(1 - \frac{1}{S_w(x)}\right)\\
&= \frac{c^2_{k-1}x^{2k-1}\big(U(x)-1\big)}{S(1)(1-x)\left[1 + c_{k-1}x^{k-1}\big(U(x)-1\big)\right]^2}
\end{aligned}$$

which is the announced result. The assumption

$$\sum_{n\geqslant 1} n^2 c_n < \infty$$

makes U twice differentiable and elementary calculations lead to

$$(\Phi^{(1)}_w)'(1) = \frac{S(1)}{c_{k-1}} - S(1) + 1 + \frac{S'(1)}{S(1)}, \qquad (\Phi^{(2)}_w)'(1) = (\Phi^{(1)}_w)'(1) + \frac{S(1)}{c_{k-1}},$$
$$(\Phi^{(1)}_w)''(1) = \frac{2S(1)^2}{c^2_{k-1}} + o\left(\frac{1}{c^2_{k-1}}\right) \quad \text{and} \quad (\Phi^{(2)}_w)''(1) = \frac{6S(1)^2}{c^2_{k-1}} + o\left(\frac{1}{c^2_{k-1}}\right),$$

and finally to (14). $\square$

4.4 Logarithmic comb and factorial comb

Let h_+ and h_- be the constants in $[0, +\infty]$ defined by

$$h_+ := \lim_{n\to+\infty} \frac{1}{n} \max\Big\{\ln\Big(\frac{1}{\pi(w)}\Big)\Big\} \text{ and } h_- := \lim_{n\to+\infty} \frac{1}{n} \min\Big\{\ln\Big(\frac{1}{\pi(w)}\Big)\Big\}, \quad (15)$$

where the maximum and the minimum range over the words w of length n with $\pi(w) > 0$. In their papers, Pittel [11] and Szpankowski [16] only deal with the cases $h_+ < +\infty$ and $h_- > 0$, which amounts to saying that the probability of any word is exponentially decreasing with its length. Here, we focus on our two examples for which these assumptions are not fulfilled. More precisely, for the logarithmic infinite comb, (2) implies that $\pi(10^n)$ is of order n^{-4}, so that

$$h_- \leqslant \lim_{n\to+\infty} \frac{1}{n} \ln\Big(\frac{1}{\pi(10^{n-1})}\Big) = 4 \lim_{n\to+\infty} \frac{\ln n}{n} = 0.$$

Besides, for the factorial infinite comb, $\pi(10^n)$ is of order $\frac{1}{(n+1)!}$ so that

$$h_+ \geqslant \lim_{n\to+\infty} \frac{1}{n} \ln\Big(\frac{1}{\pi(10^{n-1})}\Big) = \lim_{n\to+\infty} \frac{\ln n!}{n} = +\infty.$$

For these two models, the asymptotic behaviour of the lengths of the branches is not always logarithmic, as can be seen in the two following theorems, shown in Sections 4.5 and 4.6.

Theorem 4.8 (Height of the logarithmic infinite comb) *Let $\mathcal{T}_n$ be the suffix trie built from the n first suffixes of a sequence generated by a logarithmic infinite comb. Then, the height H_n of $\mathcal{T}_n$ satisfies*

$$\forall \delta > 0, \qquad \frac{H_n}{n^{\frac{1}{4}-\delta}} \xrightarrow[\mathrm{n}\to\infty]{} +\infty \quad \textit{in probability.}$$

Theorem 4.9 (Saturation level of the factorial infinite comb) *Let $\mathcal{T}_n$ be the suffix trie built from the n first suffixes of the sequence generated by a factorial infinite comb. Then, the saturation level ℓ_n of $\mathcal{T}_n$ satisfies: for any $\delta > 1$, almost surely, when n tends to infinity,*

$$\ell_n \in o\left(\frac{\ln n}{(\ln\ln n)^\delta}\right).$$

Remark 4.10 *The behaviour of the height of the factorial infinite comb can be deduced from Szpankowski [16]. Namely the property of uniform mixing stated in Proposition 3.6 implies the existence of two positive constants C and γ such that, for every integer $n \geqslant 1$ and for every word w of length n, $\pi(w) \leqslant Ce^{-\gamma n}$ (see Galves and Schmitt [8] for a proof). This entails $h_2 \geqslant \frac{\gamma}{2} > 0$ and therefore Theorem 2. in Szpankowski [16] applies: $\frac{H_n}{\ln n}$ converges almost surely to $\frac{1}{h_2}$.*
As for the saturation level of the logarithmic infinite comb, the results of [16] no longer apply since the process does not enjoy the uniform mixing property. Nevertheless, we conjecture that the saturation level has an almost sure $c \ln n$ asymptotics when n tends to infinity, for some constant c. We do not go further on this subject because it is not the main point of the present paper, but we think the techniques used here should help to prove such a result.

The dynamic asymptotics of the height and of the saturation level can be visualized on Figure 5. The number n of leaves of the suffix trie is put on the x-axis while heights or saturation levels are put on the y-axis (mean values of 25 simulations).
Plain lines illustrate our results: height of a suffix trie generated by a logarithmic comb in the first figure, saturation level of a trie generated by a factorial comb in the second one.
In the first figure, the *short dashed line* represent the height of a factorial comb. As pointed out in Remark 4.10, this height is almost surely of the form $c \ln n$ where c is a constant.
In both figures, *long dashed lines* represent a third infinite comb defined by the data $c_n = \frac{1}{3}\prod_{k=1}^{n-1}\left(\frac{1}{3} + \frac{1}{(1+k)^2}\right)$ for $n \geqslant 1$. Such a process has a uniform exponential mixing, a finite h_+ and a positive h_- as can be elementarily checked. As a matter of consequence, it satisfies all assumptions of Pittel [11] and Szpankowski [16] implying that the height and the saturation level are both of order $\ln n$. Such assumptions is fulfilled for any VLMC defined by a comb as soon as the data $(c_n)_n$ satisfy $\limsup_n c_n^{1/n} < 1$; the proof of this result is left to the reader.

These asymptotic behaviours, all coming from the same model, the infinite comb, stress its surprising richness.

Simulations of suffix tries built from a sequence generated by an infinite comb VLMC can be made by anyone on the open web site

`http://zoggy.github.com/vlmc-suffix-trie/`

This site has been designed and realized by Eng. Maxence Guesdon[6].

[6]Maxence Guesdon, `Maxence.Guesdon@inria.fr`

4.5 Height for the logarithmic comb

In this subsection, we prove Theorem 4.8.
Consider the right-infinite sequence $s = 10^\infty$. Then, $T_k(s)$ is the second occurrence time of $w = 10^{k-1}$. It is a nondecreasing (random) function of k. Moreover, $X_n(s)$ is the maximum of all k such that $s^{(k)} \in \mathcal{T}_n$. It is nondecreasing in n. So, by definition of $X_n(s)$ and $T_k(s)$, the duality can be written

$$\forall n, \forall \omega, \exists k_n, \quad k_n \leqslant X_n(s) < k_n + 1 \text{ and } T_{k_n}(s) \leqslant n < T_{k_n+1}(s). \tag{16}$$

Claim:

$$\lim_{n\to+\infty} X_n(s) = +\infty \quad \text{a.s.} \tag{17}$$

Indeed, if $X_n(s)$ were bounded above, by K say, then take $w = 10^K$ and consider $T_{K+1}(s)$ which is the time of the second occurrence of 10^K. The choice of the c_n in the definition of the logarithmic comb implies the convergence of the series $\sum_n n^2 c_n$. Thus (14) holds and $\mathbb{E}[T_{K+1}(s)] < \infty$ so that $T_{K+1}(s)$ is almost surely finite. This means that for $n > T_{K+1}(s)$, the word 10^K has been seen twice, leading to $X_n(s) \geqslant K + 1$ which is a contradiction.

We make use of the following lemma that is proven hereunder.

Lemma 4.11 *For $s = 10^\infty$,*

$$\forall \eta > 0, \quad \frac{T_k(s)}{k^{4+\eta}} \underset{\mathrm{k}\to\infty}{\longrightarrow} 0 \text{ in probability,}$$

and

$$\forall \eta > \frac{1}{2}, \quad \frac{T_k(s)}{k^{4+\eta}} \underset{k\to\infty}{\longrightarrow} 0 \quad a.s. \tag{18}$$

With notations (16), because of (17), the sequence (k_n) tends to infinity, so that $(T_{k_n}(s))$ is a subsequence of $(T_k(s))$. Thus, (18) implies that

$$\forall \eta > \frac{1}{2}, \quad \frac{T_{k_n}(s)}{k_n^{4+\eta}} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \quad \text{and} \quad \forall \eta > 0, \quad \frac{T_{k_n}(s)}{k_n^{4+\eta}} \overset{\mathrm{P}}{\underset{\mathrm{k}\to\infty}{\longrightarrow}} 0.$$

Using duality (16) again leads to

$$\forall \eta > 0, \quad \frac{X_n(s)}{n^{1/(4+\eta)}} \overset{\mathrm{P}}{\underset{\mathrm{n}\to\infty}{\longrightarrow}} +\infty.$$

In other words

$$\forall \delta > 0, \quad \frac{X_n(s)}{n^{\frac{1}{4}-\delta}} \overset{\mathrm{P}}{\underset{\mathrm{n}\to\infty}{\longrightarrow}} +\infty$$

so that, since the height of the suffix trie is larger than $X_n(s)$,

$$\forall \delta > 0, \quad \frac{H_n}{n^{\frac{1}{4}-\delta}} \xrightarrow[n\to\infty]{\mathrm{P}} +\infty.$$

This ends the proof of Theorem 4.8. □

Proof of Lemma 4.11.
Combining (13) and (14) shows that

$$\mathbb{E}(T_k(s)) = \mathbb{E}(\tau^{(2)}(w)) - k = \frac{19}{9}k^4 + o(k^4) \tag{19}$$

and

$$\mathrm{Var}(T_k(s)) = \mathrm{Var}(\tau^{(2)}(w)) = \frac{361}{162}k^8 + o(k^8). \tag{20}$$

For all $\eta > 0$, write

$$\frac{T_k(s)}{k^{4+\eta}} = \frac{T_k(s) - \mathbb{E}(T_k(s))}{k^{4+\eta}} + \frac{\mathbb{E}(T_k(s))}{k^{4+\eta}}.$$

The deterministic part in the second-hand right term goes to 0 with k thanks to (19), so that we focus on the term $\dfrac{T_k(s) - \mathbb{E}(T_k(s))}{k^{4+\eta}}$. For any $\varepsilon > 0$, because of Bienaymé-Tchebychev inequality,

$$\mathbb{P}\left(\frac{|T_k(s) - \mathbb{E}[T_k(s)]|}{k^{4+\eta}} > \varepsilon\right) \leqslant \frac{\mathrm{Var}(T_k(s))}{\varepsilon^2 k^{8+2\eta}} = \mathcal{O}\left(\frac{1}{k^{2\eta}}\right).$$

This shows the convergence in probability in Lemma 4.11. Moreover, Borel-Cantelli Lemma ensures the almost sure convergence as soon as $\eta > \frac{1}{2}$. □

Remark 4.12 *Notice that our proof shows actually that the convergence to* $+\infty$ *in Theorem 4.8 is valid a.s. (and not only in probability) as soon as* $\delta > \frac{1}{36}$.

4.6 Saturation level for the factorial comb

In this subsection, we prove Theorem 4.9.
Consider the probabilized infinite *factorial* comb defined in Section 2 by

$$\forall n \in \mathbb{N},\ c_n = \frac{1}{(n+1)!}.$$

The proof hereunder shows actually that $\left(\frac{\ell_n \ln\ln n}{\ln n}\right)_n$ is an almost surely bounded sequence, which implies the result. Recall that $\mathcal{R}$ denotes the set of all right-infinite sequences. By characterization of the saturation level as a function of X_n (see (11)), $\mathbb{P}\left(\ell_n \leqslant k\right) = \mathbb{P}\left(\exists s \in \mathcal{R},\ X_n(s) \leqslant k\right)$ for all positive integers n, k. Duality formula (12) then provides

$$\begin{aligned}\mathbb{P}\left(\ell_n \leqslant k\right) &= \mathbb{P}\left(\exists s \in \mathcal{R},\ T_k(s) \geqslant n\right)\\ &\geqslant \mathbb{P}\left(T_k(\widetilde{s}) \geqslant n\right)\end{aligned}$$

where $\widetilde{s}$ denotes any infinite word having 10^{k-1} as a prefix. Markov inequality implies

$$\forall x \in]0,1[,\ \mathbb{P}\left(\ell_n \geqslant k+1\right) \leqslant \mathbb{P}\Big(\tau^{(2)}(10^{k-1}) < n+k\Big) \leqslant \frac{\Phi^{(2)}_{10^{k-1}}(x)}{x^{n+k}} \tag{21}$$

where $\Phi^{(2)}_{10^{k-1}}(x)$ denotes as above the generating function of the rank of the final letter of the second occurrence of 10^{k-1} in the infinite random word $(U_n)_{n\geq 1}$. The simple form of the factorial comb leads to the explicit expression $U(x) = \frac{x}{(1-x)(e^x-1)}$ and, after computation,

$$\Phi^{(2)}_{10^{k-1}}(x) = \frac{e^x-1}{e-1}\cdot\frac{x^{2k-1}\big(1-e^x(1-x)\big)}{\Big[k!\,(e^x-1)\,(1-x)+x^{k-1}\big(1-e^x(1-x)\big)\Big]^2}. \tag{22}$$

In particular, applying Formulae (21) and (22) with $n=(k-1)!$ and $x = 1-\frac{1}{(k-1)!}$ implies that for any $k \geqslant 2$,

$$\mathbb{P}(\ell_{(k-1)!} \geqslant k+1) \leqslant \frac{\Big(1-\frac{1}{(k-1)!}\Big)^{2k-1}}{\Big[k!(e^{1/2}-1)\frac{1}{(k-1)!}\Big]^2}\cdot\frac{1}{\Big(1-\frac{1}{(k-1)!}\Big)^{(k-1)!+k}}.$$

Consequently, $\mathbb{P}\left(\ell_{(k-1)!} \geqslant k+1\right) = \mathcal{O}(k^{-2})$ is the general term of a convergent series. Thanks to Borel-Cantelli Lemma, one gets almost surely

$$\overline{\lim_{n\to+\infty}}\ \frac{\ell_{n!}}{n} \leqslant 1.$$

Let Γ^{-1} denote the inverse of Euler's Gamma function, defined and increasing on the real interval $[2,+\infty[$. If n and k are integers such that $(k+1)! \leqslant n \leqslant (k+2)!$, then

$$\frac{\ell_n}{\Gamma^{-1}(n)} \leqslant \frac{\ell_{(k+2)!}}{\Gamma^{-1}((k+1)!)} = \frac{\ell_{(k+2)!}}{k+2},$$

which implies that, almost surely,

$$\overline{\lim_{n\to\infty}}\, \frac{\ell_n}{\Gamma^{-1}(n)} \leqslant 1.$$

Inverting Stirling Formula, namely

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} e^{x\log x - x}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right)$$

when x goes to infinity, leads to the equivalent

$$\Gamma^{-1}(x) \underset{+\infty}{\sim} \frac{\log x}{\log\log x},$$

which implies the result. $\square$

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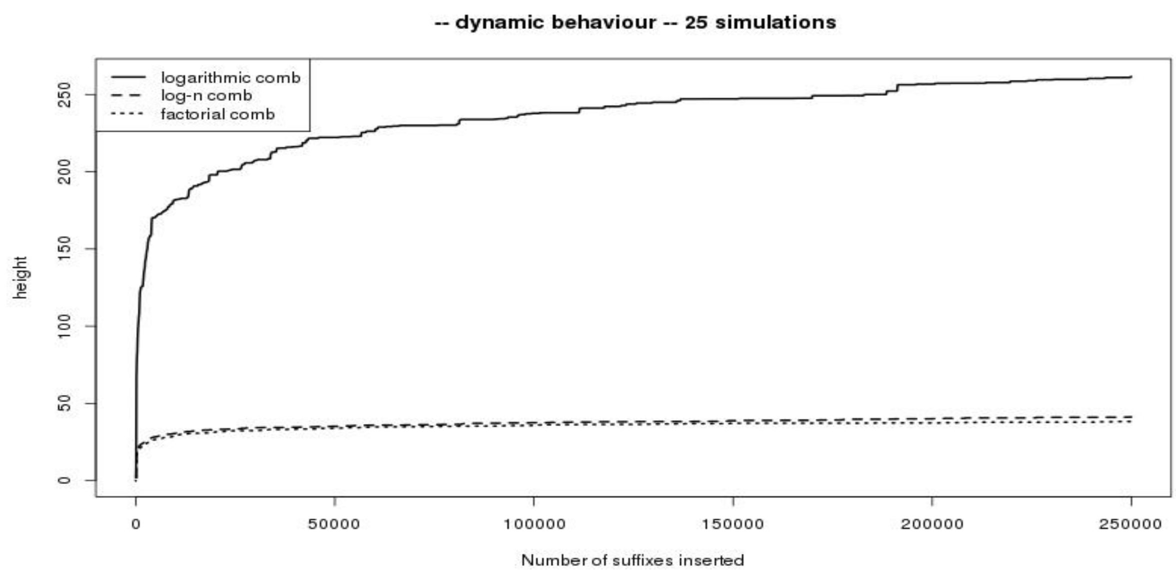

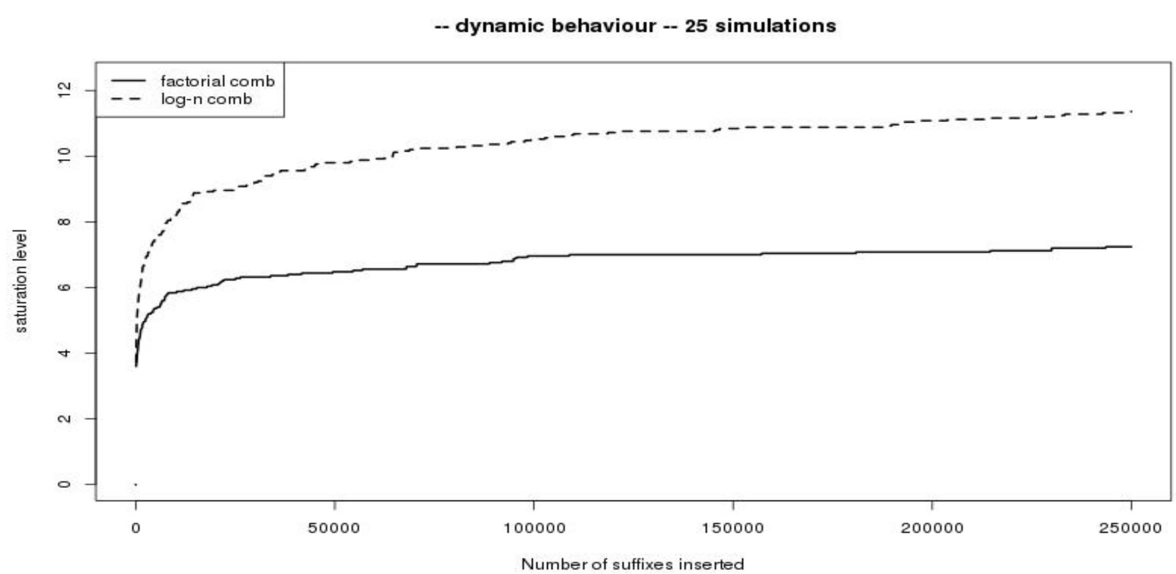

Figure 5: Above: the height of a logarithmic comb (plain lines) compared with the height of a $\ln n$-comb (long dashed lines) and with the height of a factorial comb (short dashed lines). Below: the saturation level of a factorial comb (plain lines) compared with the saturation level of a $\ln n$-comb (long dashed lines). In both figures, mean values of 25 simulations are represented on the y-axes.